Report at annual TORUS Collaboration meeting

Vasily Eremenko[1]

[1]Ohio University, Dept. of Physics & Astronomy

Livermore, CA
2013-06-11

Introduction

The “formalism article”

A. M. Mukhamedzhanov et al. // Phys. Rev. C **86**, 034001 (2012).

Introduction

The "formalism article"

A. M. Mukhamedzhanov et al. // Phys. Rev. C **86**, 034001 (2012).

The first step

is to compute the (separable) nuclear potential matrix elements in Coulomb basis.

$$\langle p^C|V|p^C_\alpha\rangle = Z^{SC}(p^C, p^C_\alpha)$$
$$= \frac{1}{(2\pi)^4}\left[\int dk\; k^2(\psi^C_{lp}(k))^* h_{lp}(k)\right]\lambda\left[\int d\kappa\; \kappa^2 h^*_{lp_\alpha}(\kappa)\psi^C_{lp_\alpha}(\kappa)\right]. \quad (1)$$

Computing matrix element...

It requires:

- Form-factors $h_{lp_\alpha}(k)$ (Yamaguchi-style $\longrightarrow$ the EST potentials).
- Coulomb wave function in momentum space $\psi^C_{lp}(k)$.
  - Subroutines to compute ${}_2F_1(a,b;c;z)$ and $\Gamma(z)$ for a complex arguments.
- $\psi^C_{lp_\alpha}(k)$ has a singularity $S(k-p_\alpha) = (k-p_\alpha)^{-1-i\eta}$ at $k = p_\alpha$.
  $\Longrightarrow$ Integration algorithm (with regularization, proofs, etc.).

Coulomb wave function $\psi^C_{lp_\alpha}(k)$

Special functions

Now we have the tested, reliable and fast enough implementations of

- $\Gamma(x)$, $\Gamma(z)$,
- ${}_2F_1(a, b; c; \xi)$,

for $a, b, c, z \in \mathbb{C}$ and $\xi, x \in \mathbb{R}$.

Coulomb wave function $\psi^C_{lp_\alpha}(k)$

Special functions

Now we have the tested, reliable and fast enough implementations of

- $\Gamma(x)$, $\Gamma(z)$,
- ${}_2F_1(a, b; c; \xi)$,

for $a, b, c, z \in \mathbb{C}$ and $\xi, x \in \mathbb{R}$.

Hypergeometric function

is well-defined, if $|\xi| < 1$ (other requirements are omited).

Coulomb wave function $\psi^C_{lp_\alpha}(k)$ (cont.)

Representations

There are two representations of $\psi^C_{lp_\alpha}(k)$

- the *'pole'* representation (works well near the pole $k = p_\alpha$),
- and the *'non-pole'* one.

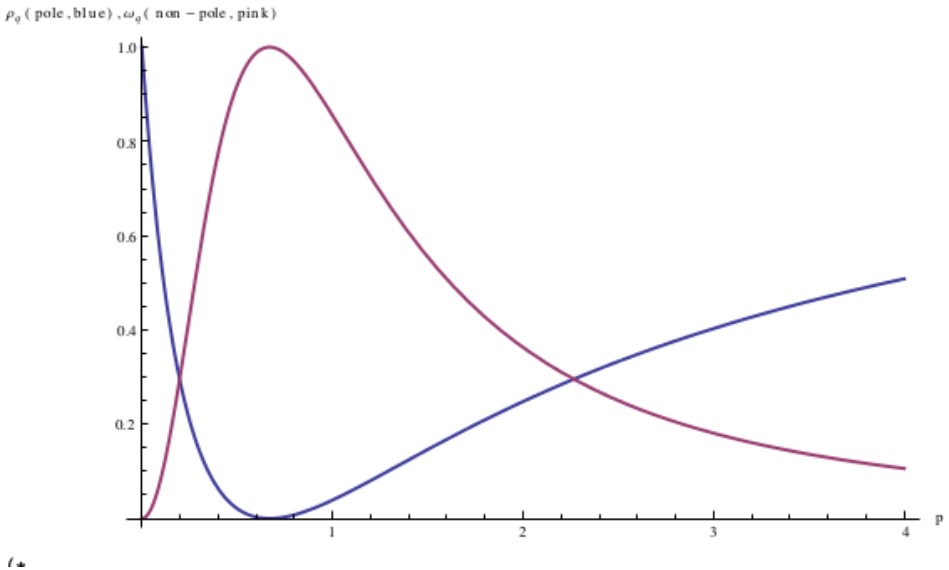
Coulomb wave function $\psi^C_{lp_\alpha}(k)$ (cont.)

Representations

There are two representations of $\psi^C_{lp_\alpha}(k)$

- the *'pole'* representation (works well near the pole $k = p_\alpha$),
- and the *'non-pole'* one.

The differences are mostly at the 4-th argument of hypergeometric function



Coulomb wave function $\psi^C_{lp_\alpha}(k)$ (cont.)

Switching between representations

The previously shown 4-th arguments

- are equal one another at the points

$$\text{pMin} = 0.3p_\alpha; \qquad \text{pMax} = 3p_\alpha; \tag{2}$$

- and both are about 0.3 at pMin and pMax.

Coulomb wave function $\psi^C_{lp_\alpha}(k)$ (cont.)

Switching between representations

The previously shown 4-th arguments

- are equal one another at the points

$$\text{pMin} = 0.3p_\alpha; \qquad \text{pMax} = 3p_\alpha; \tag{2}$$

- and both are about 0.3 at pMin and pMax.

Suggestion

If pMin $< k <$ pMax, $\Rightarrow$ use 'pole' representation of $\psi^C_{lp_\alpha}(k)$;
and 'none-pole' representation otherwise.
In this case we shall be at the well-defined region for any reasonable k and p_α. The choice will be done very fast (very simple formulas to compute).

Computing matrix element... (cont.)

$$\langle p^C|V|p^C_\alpha\rangle = Z^{SC}(p^C, p^C_\alpha)$$
$$= \frac{1}{(2\pi)^4}\left[\int dk\; k^2(\psi^C_{lp}(k))^* h_{lp}(k)\right]\lambda\left[\int d\kappa\; \kappa^2 h^*_{lp_\alpha}(\kappa)\psi^C_{lp_\alpha}(\kappa)\right]. \quad (3)$$

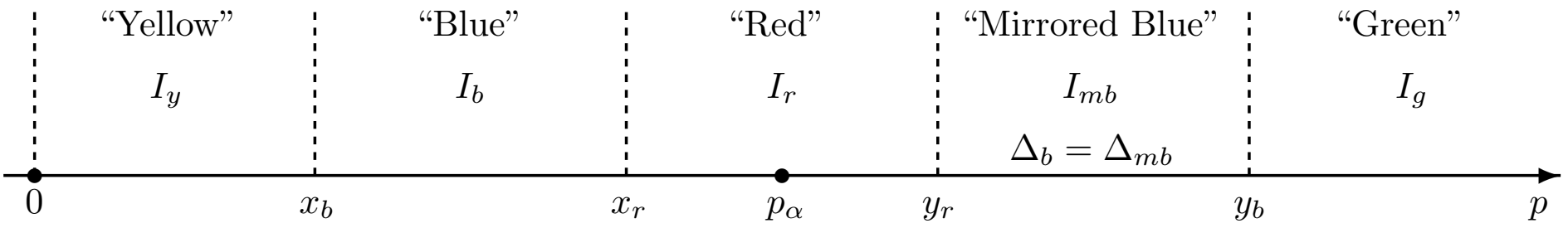
Coulomb distorted potential matrix element requires:

- Form-factors $h_{lp_\alpha}(k)$ (Yamaguchi-style $\longrightarrow$ the EST potentials).
- Coulomb wave function in momentum space $\psi^C_{lp}(k)$.
  - Subroutines to compute ${}_2F_1(a,b;c;z)$ and $\Gamma(z)$ for a complex arguments.
- $\psi^C_{lp_\alpha}(k)$ has a singularity $S(k-p_\alpha) = (k-p_\alpha)^{-1-i\eta}$ at $k = p_\alpha$.
  $\Longrightarrow$ Integration algorithm (with regularization, proofs, etc.).

Integration algorithm

'Colored' regions

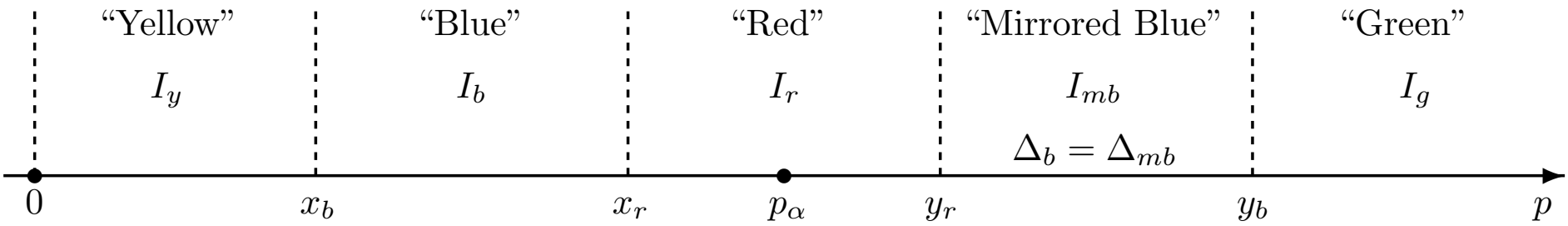
Basing on the behavior of the $\psi^C_{lp_\alpha}(p)$ (oscillations of leading singularity)



Integration algorithm

'Colored' regions

Basing on the behavior of the $\psi^C_{lp_\alpha}(p)$ (oscillations of leading singularity)



Estimations for the borders are

$$x_b = p_\alpha - 0.1/[1 - \exp(-2\pi/\eta_\alpha)] \sim p_\alpha - 0.1 \text{ fm}^{-1}; \quad (4)$$

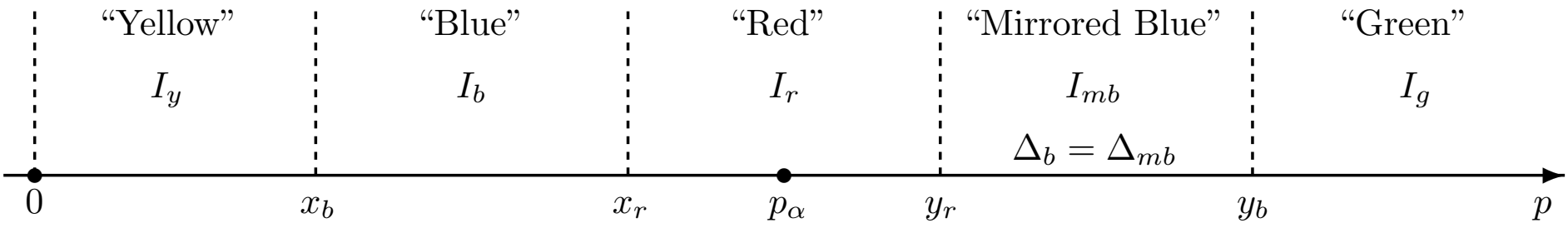
$$x_r = p_\alpha - 10^{-6}/[1 - \exp(-2\pi/\eta_\alpha)] \sim p_\alpha - 10^{-6} \text{ fm}^{-1}; \quad (5)$$

and y_r and y_b are symmetric to corresponding x_i.

Integration algorithm (cont.)

Suitable tools to be used

Just to be in sight



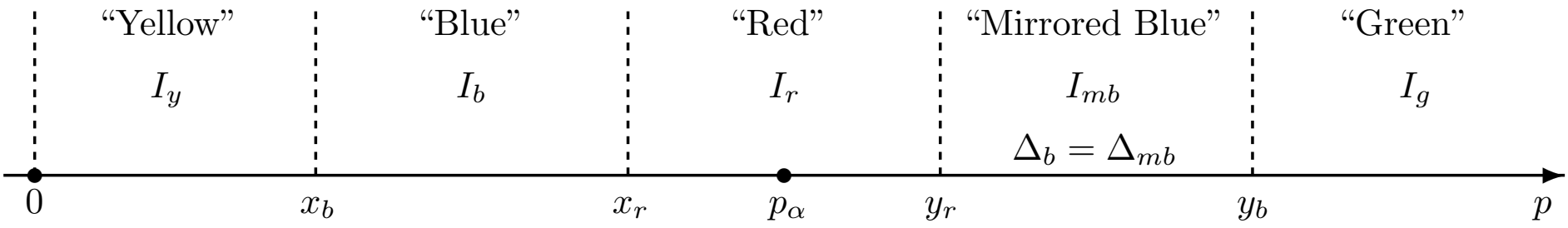
Gauss Quagratures

at the 'yellow' and 'green' regions.

Integration algorithm (cont.)

Suitable tools to be used

Just to be in sight



Gauss Quagratures

at the 'yellow' and 'green' regions.

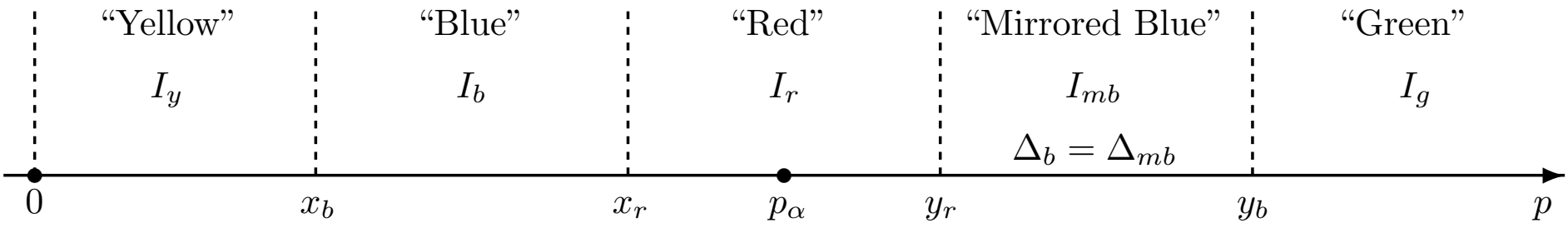
Simpson rule

at the 'blue' and 'mirrored blue' regions.

Integration algorithm (cont.)

Suitable tools to be used

Just to be in sight



Gauss Quagratures

at the 'yellow' and 'green' regions.

Simpson rule

at the 'blue' and 'mirrored blue' regions.

Regularization + analytical approximations

at the 'red' region.

Integration algorithm (cont.)

Suitable tools to be used (cont.)

The outcome:

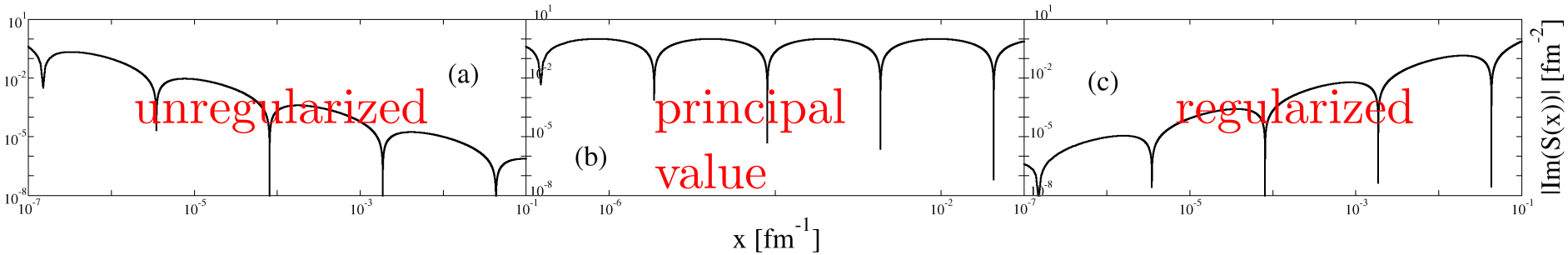
we are able to compute all the integral alone the positive real axis of momentum. No *terra incognita*.

Integration algorithm (cont.)

Integration over the 'red' region

Regularization

Well-known 'principal value' regularization is not enough.
Gel'fand-Shilov regularization is required.



Integration algorithm (cont.)

Integration over the 'red' region (cont.)

Ron Johnson's suggestion

Because $h_{lp_\alpha}(k)$ is slowly variable function at the 'red' region, and $\psi^C_{lp_\alpha}(p)$ is oscillating wildly, mean-value theorem could be applied:

$$\int_{red} dk\ k^2 \psi^C_{lp_\alpha}(k) h^*_{lp_\alpha}(k) = h^*_{lp_\alpha}(p_\alpha) \int_{red} dk\ k^2 \psi^C_{lp_\alpha}(k). \tag{6}$$

Integration algorithm (cont.)

Integration over the 'red' region (cont.)

Ron Johnson's suggestion

Because $h_{lp_\alpha}(k)$ is slowly variable function at the 'red' region, and $\psi^C_{lp_\alpha}(p)$ is oscillating wildly, mean-value theorem could be applied:

$$\int_{red} dk\ k^2 \psi^C_{lp_\alpha}(k) h^*_{lp_\alpha}(k) = h^*_{lp_\alpha}(p_\alpha) \int_{red} dk\ k^2 \psi^C_{lp_\alpha}(k). \tag{6}$$

Now regularizing the rest

After applying the regularization, in case of small Δ_{red},

$$I_r \approx \frac{h^*_{lp_\alpha} e^{\pi\eta_\alpha} \phi''(p_\alpha)}{2(2 - i\eta_\alpha)} \Delta_{red}^{2-i\eta_\alpha}, \tag{7}$$

where $\phi(k)$ is the rest of $k^2\psi^C_{lp_\alpha}(k)$ after pulling out the leading singularity $S(k - p_\alpha) = (k - p_\alpha)^{-1-i\eta}$.

Integration algorithm (cont.)

Relative contributions of the regions

Using Mathematica

In case of $p + {}^{40}\mathrm{Ca}$ @ $E_{cm} \approx 10$ MeV ($\eta = 1$, $p_\alpha = 0.67$ fm^{-1}) with Yamaguchi-style potential ($\kappa = 2.7$ fm^{-1}).

$$\Delta_{blue} = |p_\alpha - x_b| = |p_\alpha - y_b| = 0.1 \text{ fm}^{-1}; \tag{8}$$

$$\Delta_{red} = |p_\alpha - x_r| = |p_\alpha - y_r| = 1 \cdot 10^{-6} \text{ fm}^{-1}. \tag{9}$$

Integration algorithm (cont.)

Relative contributions of the regions

Using Mathematica

In case of $p + {}^{40}\mathrm{Ca}$ @ $E_{cm} \approx 10$ MeV ($\eta = 1$, $p_\alpha = 0.67$ fm^{-1}) with Yamaguchi-style potential ($\kappa = 2.7$ fm^{-1}).

$$\Delta_{blue} = |p_\alpha - x_b| = |p_\alpha - y_b| = 0.1 \text{ fm}^{-1}; \tag{8}$$

$$\Delta_{red} = |p_\alpha - x_r| = |p_\alpha - y_r| = 1 \cdot 10^{-6} \text{ fm}^{-1}. \tag{9}$$

Integral	**relative contribution**
I_y	-3.11885
I_b	2.31924
I_r	$2.4 \cdot 10^{-8}$ (near negligible)
I_{mb}	-0.115053
I_g	0.355378
Result:	-0.559282

Summary (was done + To-Do)

- The theory is ready to compute $\psi_{lp}^{C}(k)$ and Coulomb distorted EST potential matrix element $\langle p^{C}|V|p_{\alpha}^{C}\rangle$.
- Fortran implementation of the discussed algorithms is on the way.
- Specifically, the following is still waiting to be implemented:
  - 'intelligent' $\psi_{lp}^{C}(k)$,
  - integration algorithm to compute $\langle p^{C}|V|p_{\alpha}^{C}\rangle$.
- Using the obtained code, compute $\psi_{lp}^{C}(k)$ and $\langle p^{C}|V|p_{\alpha}^{C}\rangle$ for different cases.
- Publish it.